Periodic Solutions of Second Order Differential Equations

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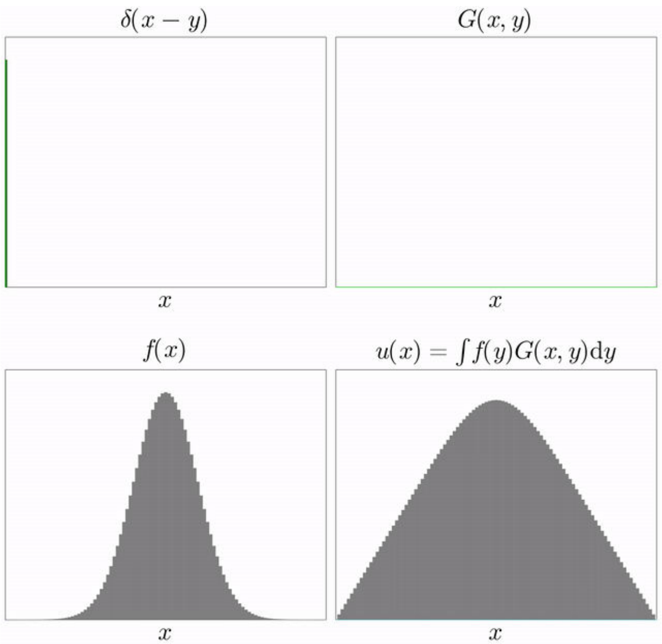
Abstract

We study the existence of solutions to the second order differential equation through topological degree theory.
In this poster I show the **Brouwer Fixed Point Theorem** which states that if $f \in C(\overline{D}, \overline{D})$ where D is open, then there exists a $x \in D$ such that $f(x) = x$.

Background Information

The Dirac Delta can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite and it satisfies the following two conditions $\int_{-\infty}^{\infty} \delta(t)dt = 1$ and $\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$.

A Green's function, $G(x, s)$, of a linear differential operator $\mathcal{L} = \mathcal{L}(x)$ acting at a point s, is any solution of $\mathcal{L}G(x, s) = \delta(s - x)$. The reason as to why Greens function is considered is as given a ODE, L(solution) = source, when we first solve $\mathcal{L}$(Green) $= \delta_s$, for each s, and realizing that, since the source is a sum of delta functions, the solution is a sum of Green's functions as well, by linearity of $\mathcal{L}$.



Solution of $x'' = f(t, x, x')$

We first find the Green Function for $x'' + \lambda^2 x = \delta(t - s)$,

$$G(t, s) = \begin{cases} \dfrac{\sin(\lambda(t - s + \omega)) + \sin(\lambda(s - t))}{2\lambda(1 - \cos(\lambda\omega))} & 0 \le t < s \le \omega \\ \dfrac{\sin(\lambda(s - t + \omega)) + \sin(\lambda(t - s))}{2\lambda(1 - \cos(\lambda\omega))} & 0 \le s < t \le \omega \end{cases}$$

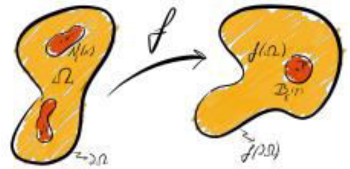
This is verified as it satisfies the continuity, derivative jump and symmetric properties of Greens functions. Now since $G(t, s)$ is a solution for the above ODE we claim the following.

Claim 1. *The solution to the following B.V.P,*

$$\begin{cases} x'' = f(t, x(t), x'(t)) \\ x(0) = x(\omega) \\ x'(0) = x'(\omega) \end{cases}$$

is $x(t) = \int_0^{\omega} G(t, s) \{f(s, x(s), x'(s)) + \lambda^2 x(s)\} ds$

Brouwer Degree



If $U \subset \mathbb{R}^n$ is a bounded region, $f : \overline{U} \to \mathbb{R}^n$ smooth, p a regular value of f and $p \notin f(\partial U)$, then the degree deg(f, U, p) is defined by the formula

$$\deg(f, U, p) := \sum_{x_i \in f^{-1}(p)} \operatorname{sgn} \det(J_f(x_i))$$

The degree satisfies the following properties:
(p1) (normalization) If $0 \in U$, then $\deg(I, U) = 1$;
(p2) (additivity) Let $U_1, U_2 \subset U$ be such open sets that $U_1 \cap U_2 = \emptyset$ and $0 \notin f\left(\overline{U} \backslash (U_1 \cup U_2)\right)$, then

$$\deg(f, U) = \deg\left(f|_{\overline{U_1}}, U_1\right) + \deg\left(f|_{\overline{U_2}}, U_2\right)$$

(p3) (homotopy invariance) if f and g are homotopy equivalent via a homotopy f_t such that $f_0 = f, f_1 = g$ and $p \notin f_t(\partial\Omega)$ then $\deg(f, U) = \deg(g, U)$.

Properties and Results

Here we assume f is defined as before.
(p4) $\deg(f, \emptyset) = 0$
(p5) (excision) Let $V \subset U$ is such open bounded set that $0 \notin f(\overline{U} \backslash V)$. Then $\deg(f, U) = \deg\left(f|_{\overline{V}}, V\right)$
(p6)Let f be defined as before be such that $0 \notin f(\overline{U})$. Then $\deg(f, U) = 0$.
(p7) (existence). Assume $\deg(f, U) \neq 0$. Then there exists such $x_0 \in U$, that $f(x_0) = 0$

We call a point **regular** if $J_f(x) \neq 0$ whenever $x \in U$ and $f(x) = 0$.

Lemma 2. *The set of regular points* $f^{-1}(\{0\}) = \{x_1, x_2, \ldots x_N\}$ *is finite.*

Let $h : \mathbb{R}^N \to \mathbb{R}$ is a smooth function such that $\int_{\mathbb{R}^N} h(x)dx = 1$, and $h(x) = 0$ outside of a ball $B_\varepsilon(0)$ for some small $\varepsilon > 0$.

Integral Representation

$$\sum_{x_i \in f^{-1}(0)} \operatorname{sign}(det(J_f(x_i))) = \int_U h(f(x))det(J_f(x))dx$$

where h, f, U are defined as before. In fact, the above integral is independent of h!

Brouwer Fixed Point Theorem

Lemma 3. *Let $f \in C^1(U) \cap C(\overline{U})$, U is the open unit ball and $f(x).x > 0 \; \forall \, x \in \partial U$. Show $\exists \, c \in U \ni f(c) = 0$.*

Theorem
Let $f \in C^1(D) \cap C(\overline{D})$, $f : D \to D \implies \exists \, c \in D \ni f(c) = c$.

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References

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